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On C^∞-Vectors and Intertwining Bilinear Forms for Representations of Lie Groups

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The main result of this paper is a structure theorem for continuous intertwining bilinear forms on the Fréchet spaces of C^∞-vectors of two Banach space representations of a Lie group. Using elliptic regularity theory it is shown that such forms can be identified with a certain class of closed densely defined intertwining operators. As an application of this result it is shown that all the usual criteria for equivalence and irreducibility of unitary representations remain valid for the corresponding differentiable representations on the spaces of C^∞-vectors. The results are used to study families of representations having a common space of C^∞-vectors, and the theory is illustrated by some examples. The spaces of C^∞-vectors of the regular representations in $L^p(G)$ are described, and the result is used to characterize a certain class of closed translation invariant operators from $L^p(G)$ to $L^q(G)$. The same technique is used to characterize the space of C^∞-vectors of an arbitrary induced representation of a Lie group.

INTRODUCTION

Let U be a continuous unitary representation of a Lie group G in a Hilbert space $\mathbf{H}$. Let $\beta(\cdot, \cdot)$ be a continuous sesquilinear form on the Fréchet space $\mathbf{D}_\infty(U)$ of C^∞-vectors for U such that

$$\beta(U(g)x, U(g)y) = \beta(x, y) \qquad \text{for all} \quad g \in G, \quad x, y \in \mathbf{D}_\infty(U).$$

If U is irreducible we show that $\beta(\cdot, \cdot)$ has to be a multiple of the scalar product on $\mathbf{H}$:

$$\beta(x, y) = \text{const}\langle x, y\rangle \qquad \text{for all} \quad x, y \in \mathbf{D}_\infty(U).$$

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On the other hand, assuming that the scalar product is essentially the only continuous group invariant sesquilinear form on $\mathbf{D}_\infty(U)$, it is easy to show that U is irreducible in $\mathbf{H}$ (Corollary 3.4).

In [28, Section 3] Segal proved an analogous result for the action of a quantum process on the space of C^∞-vectors for the energy operator. The result of Corollary 3.4 was conjectured by Segal in [28] on the basis of the many similarities between quantum field theory and the theory of group representations.

If the representation U is not irreducible it is still possible to give a nice description of all continuous group invariant sesquilinear forms on $\mathbf{D}_\infty(U)$. Specifically, we prove that β has the form $\beta(x, y) = \langle Tx, y\rangle$, where T is a group invariant closed linear operator in H which maps $D_\infty(U)$ into itself continuously (Corollary 2.1). The operator T is unique, and in case T is formally normal on $D_\infty(U)$ we show that T is automatically normal (Corollary 2.4).

The proof is based on elliptic regularity theory, and our methods allow us to study continuous intertwining bilinear forms on the spaces of C^∞-vectors of two arbitrary Banach space representations. As the main result of Section 2 we establish a bijective correspondence between such forms and certain closed densely defined intertwining operators (see also the introduction of Section 2).

Section 1 contains some general results on C^∞-vectors for a representation of a Lie group. Using a method due to Goodman we characterize the space of C^∞-vectors of the contragredient representation. We prove a general density theorem (Theorem 1.3) which is of some independent interest because it shows that for many purposes the choice of (group invariant) domain is immaterial (see e.g. Corollary 1.2). As a simple application of this result we identify the infinitesimal generator of the heat equation semigroup constructed by Nelson.

In Section 3 we prove some results on irreducibility and equivalence. Originally, it was the author's hope that a detailed structure theorem for intertwining bilinear forms would lead to a better notion of (weak) equivalence of representations of Lie groups. We give an example which shows that this hope was too optimistic (even for representations in Hilbert spaces). For unitary representations the results of Section 2 give new criteria for irreducibility and equivalence. We show that all the usual Hilbert space conditions remain valid for the associated representations on the Fréchet spaces of C^∞ vectors.

In Section 4 we study families of Banach space representations having a common space of C^∞-vectors. In case the family contains an irreducible unitary representation we get some additional information about various types of irreducibility and equivalence of all representations in the family.

Section 5 contains some examples. First we consider various Banach space representations of the Heisenberg group to illustrate the theory developed in Section 4. We characterize the space of C^∞-vectors for the regular representations of a Lie group G in $L^p(G)$. The result is used to describe a certain class of translation invariant operators from $L^p(G)$ to $L^q(G)$.

Using elliptic operators Blattner proved that a C^∞-vector for an induced representation U^L (of a Lie group) is a continuous function in case the representation L of the subgroup is finite dimensional. In Section 5 we use our density theorem to give a complete characterization of the space of C^∞-vectors of an arbitrary induced representation. In particular, we prove that a C^∞-vector is actually an infinitely differentiable function, and an analytic vector is an analytic function. Also, we establish the fact that point evaluation always defines a continuous linear mapping on the space of C^∞-vectors.

The results of the present paper are essentially contained in the author's Ph. D. Thesis [21], written under direction of Professor I. E. Segal. The author is indebted to Professor Segal for many helpful discussions on the theory of group representations.

1. General Results on C^∞ Vectors

In order to establish the notation it is convenient to recall some results on representation theory [5, 11, 20, 25].

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $g \to V(g)$ be a strongly continuous representation of G in a Banach space $\mathbf{B}$. A vector $x \in \mathbf{B}$ is called a C^∞ vector for V if the mapping $g \to V(g)x$ is C^∞ from G to $\mathbf{B}$ or equivalently if the function $g \to \langle V(g)x, x^*\rangle$ is C^∞ on G for each continuous linear functional $x^* \in \mathbf{B}^*$. The set of C^∞ vectors is clearly a linear subspace of $\mathbf{B}$ which we will denote by $\mathbf{D}_\infty$ or $\mathbf{D}_\infty(V)$.

On $\mathbf{D}_\infty$ we have a representation v of $\mathfrak{g}$ defined by

$$v(X)x = \frac{d}{dt} V(\exp(tX))x \Big|_{t=0} \qquad \text{for} \qquad X \in \mathfrak{g}, \quad x \in \mathbf{D}_\infty .$$

The mapping $X \to v(X)$ is a representation of $\mathfrak{g}$ as a Lie algebra of operators having $\mathbf{D}_\infty$ as a common invariant dense domain, and v has a unique extension to a representation, also denoted by v, of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$.

Let $\{X_1, X_2, \ldots, X_d\}$ be a basis for $\mathfrak{g}$, and let $v_1(X_k)$ denote the infinitesimal generator of the one-parameter group $t \to V(\exp(tX_k))$. Then $\mathbf{D}_\infty$ can be characterized in the following way [11, Theorem 1.1]

$$\mathbf{D}_\infty = \bigcap_{k=1}^{d} \bigcap_{n=1}^{\infty} \mathbf{D}_{v_1(X_k)^n},$$

where $\mathbf{D}_{v_1(X_k)^n}$ denotes the domain of $v_1(X_k)^n$. In particular, $\mathbf{D}_\infty$ coincides with "the maximal domain for V" employed by Segal [26, 27].

Following Goodman [11] we topologize $\mathbf{D}_\infty$ by the following family of seminorms ρ_n:

$$\rho_n(x) = \sum_{1 \leqslant i_k \leqslant d} \| v(X_{i_1} \cdots X_{i_n})x \|$$

for $n = 0, 1, 2\ldots$ (with the interpretation $\rho_0(x) = \| x \|$). Then $\mathbf{D}_\infty$ is a Fréchet space [11] and for $g \in G$ the restriction $V_\infty(g)$ of $V(g)$ to $\mathbf{D}_\infty$ is a continuous linear operator on $\mathbf{D}_\infty$. Using the relation

$$g \cdot \exp(tX) \cdot g^{-1} = \exp(Ad(g) \cdot tX)$$

this can be seen directly, but it is also an immediate consequence of the closed graph theorem.

For $x \in \mathbf{B}$ and $\phi \in C_0^\infty(G)$ we let

$$V(\phi)x = \int_G \phi(a)\, V(a)x \, da,$$

where da is some left invariant Haar measure on G. Then $V(\phi)x$ is a C^∞ vector for V, and we have

$$v(X)\, V(\phi)x = V(X\phi)x,$$

where

$$(X\phi)(a) = \frac{d}{dt} \phi(\exp(-tX) \cdot a) \Big|_{t=0}$$

for $a \in G$.

The linear subspace spanned by vectors of the form $V(\phi)x$ ($x \in \mathbf{B}$, $\phi \in C_0^\infty(G)$) is usually called the Gårding space for V, and $\mathbf{D}_\infty(V)$ can also be described as the completion of this space (cf. Theorem 1.3 below).

Now let $\hat{V}$ denote the contragredient representation of V in the sense of Bruhat [5, p. 113]; i.e., $\hat{V}(g) = V(g^{-1})^* |_{\hat{\mathbf{B}}}$ where $\hat{\mathbf{B}}$ is the (strongly closed) linear subspace of elements $x^* \in \mathbf{B}^*$ for which the mapping $g \to V(g^{-1})^* x^*$ is strongly continuous. For $X \in \mathfrak{g}$ we let $\hat{v}_1(X)$ denote the infinitesimal generator of the one-parameter group $t \to \hat{V}(\exp(tX))$. The proof of the following simple result is left to the reader.

LEMMA 1.1. *For all $X \in \mathfrak{g}$ we have $\mathbf{D}_{\hat{v}_1(X)} = \{\hat{x} \in \hat{\mathbf{B}} \mid \hat{x} \in \mathbf{D}_{v_1(X)^*}$ and $v_1(X)^* \hat{x} \in \hat{\mathbf{B}}\}$ and $\hat{v}_1(X)\hat{x} = -v_1(X)^* \hat{x}$ for $\hat{x} \in \mathbf{D}_{\hat{v}_1(X)}$.*

Let $\hat{v}$ denote the associated representation of $\mathfrak{U}(\mathfrak{g})$ on $\mathbf{D}_\infty(\hat{V})$. Then for $x \in \mathbf{D}_\infty(V)$, $\hat{x} \in \mathbf{D}_\infty(\hat{V})$, and $L \in \mathfrak{U}(\mathfrak{g})$ we get

$$\langle v(L)x, \hat{x} \rangle = \langle x, \hat{v}(L^*)\hat{x} \rangle,$$

where $L \to L^*$ denotes the usual *-operation in $\mathfrak{U}(\mathfrak{g})$. In particular, we note that $v(L)^* \supseteq \hat{v}(L^*)$, and since $\mathbf{D}_\infty(\hat{V})$ is w^*-dense in $\mathbf{B}^*$ [5] it follows that $v(L)$ is closable for each $L \in \mathfrak{U}(\mathfrak{g})$.

In order to characterize the C^∞ vectors for $\hat{V}$ we need the following lemma.

LEMMA 1.2. *Let $I \subseteq \mathbb{R}$ be an open subset and let f be a mapping from I into $\mathbf{B}^*$ such that the function $t \to \langle x, f(t) \rangle$ is of class $C^2(I)$ for each $x \in \mathbf{B}$. Then f is of class C^1 in the norm topology.*

Proof. For each $t \in I$ there exists an element $f'(t) \in \mathbf{B}^*$ such that

$$\langle x, f'(t) \rangle = \lim_{h \to 0} \langle x, h^{-1}[f(t+h) - f(t)] \rangle$$

for all $x \in \mathbf{B}$. By the principle of uniform boundedness

$$\|f(t+h) - f(t)\| \leqslant \text{const} \mid h \mid$$

for $\mid h \mid$ sufficiently small. Hence f is strongly continuous, and the same argument shows that the mapping $t \to f'(t)$ is strongly continuous. We have to show that f' is also the derivative in the norm topology. For $x \in \mathbf{B}$ we have

$$\langle x, h^{-1}[f(t+h) - f(t)] - f'(t) \rangle = \frac{1}{h} \int_t^{t+h} \langle x, f'(s) - f'(t) \rangle \, ds$$

for $\mid h \mid$ sufficiently small. Hence (for $h > 0$)

$$\begin{aligned} &\| h^{-1}[f(t+h) - f(t)] - f'(t) \| \\ &\qquad \leqslant \frac{1}{h} \int_t^{t+h} \|f'(s) - f'(t)\| \, ds \to 0 \qquad \text{as} \qquad h \to 0, \end{aligned}$$

since the integrand is continuous. Q.E.D.

PROPOSITION 1.1. *Let $x^* \in \mathbf{B}^*$. Then x^* is a C^∞ vector for $\hat{V}$ iff the mapping $g \to \langle V(g)x, x^*\rangle$ is C^∞ for each $x \in \mathbf{B}$.*

Proof. Let $\{X_1, \ldots, X_d\}$ be a basis for $\mathfrak{g}$ and let

$$g(t) = \exp(t_1X_1) \cdots \exp(t_dX_d) \qquad \text{for} \quad t = (t_1, t_2, \ldots, t_d) \in \mathbb{R}^d.$$

Then, the mapping $g(t) \to t$ is an analytic coordinate system in a neighborhood of e in G [12, Ch. II]. For $t_k \in \mathbb{R}$ we let $F_k(t_k) = V(\exp(-t_kX_k))^*$. Suppose $x^* \in \mathbf{B}^*$ has the property that $g \to \langle V(g)x, x^*\rangle$ is C^∞ for each $x \in \mathbf{B}$. If $K \subseteq \mathbb{R}^d$ is a compact neighborhood of 0 we have

$$M = \sup_{t \in K} \| V(g(t)^{-1})^* \| < \infty,$$

and

$$\begin{aligned} &\| V(g(t)^{-1})^* x^* - x^* \| \\ &\quad \leqslant \| F_1(t_1) \cdots F_{d-1}(t_{d-1})[F_d(t_d)\, x^* - x^*]\| \\ &\qquad + \| F_1(t_1) \cdots F_{d-2}(t_{d-2})[F_{d-1}(t_{d-1})\, x^* - x^*]\| \cdots + \| F_1(t_1)\, x^* - x^* \| \\ &\quad \leqslant M \sum_{k=1}^{d} \| F_k(t_k)\, x^* - x^* \| \qquad \text{for all} \quad t \in K. \end{aligned}$$

By Lemma 1.2 the mapping $t_k \to F_k(t_k)x^*$ is C^∞ in the norm topology; so $x^* \in \hat{\mathbf{B}}$. Then $x^* \in \mathbf{D}_\infty(\hat{V})$ by Lemma 1.1 and Goodman's characterization of the C^∞ vectors. Q.E.D.

THEOREM 1.1. *Let $L \in \mathfrak{U}(\mathfrak{g})$ be elliptic and let $A = \overline{v(L)}$. Then*

$$\mathbf{D}_\infty(\hat{V}) = \bigcap_{n=0}^{\infty} \mathbf{D}_{(A^*)^n}$$

Proof. Suppose $x \in \mathbf{B}$ and $x^* \in \mathbf{D}_{(A^*)^n}$ for all $n \in \mathbb{N}$. We want to show that the function $f(a) = \langle V(a)x, x^*\rangle$, $a \in G$ is a weak solution of an elliptic equation of arbitrarily high order. For $\phi \in C_0^\infty(G)$ we have

$$\begin{aligned} \int_G (L^n\phi)(a) f(a)\, da &= \langle V(L^n\phi)x, x^*\rangle = \langle A^n V(\phi)x, x^*\rangle \\ &= \langle V(\phi)x, (A^*)^n x^*\rangle = \int_G \phi(a) f_n(a)\, da, \end{aligned}$$

where $f_n(a) = \langle V(a)x, (A^*)^n x^*\rangle$ for $a \in G$. Since each f_n is continuous it follows from the elliptic regularity theorem [2, 190] that f is C^∞ on G. Then by Proposition 1.1 $x^* \in \mathbf{D}_\infty(\hat{V})$. Q.E.D.

Similar arguments give the following result.

COROLLARY 1.1. *If $\{X_1, \ldots, X_d\}$ is a basis for* $\mathfrak{g}$, *then*

$$\mathbf{D}_\infty(\hat{V}) = \bigcap_{k=1}^{d} \bigcap_{n=1}^{\infty} \mathbf{D}_{(v_1(X_k)^*)^n}.$$

Now we return to the study of the representation V. In Section 2 we shall find it convenient to use a different description of the topology on $\mathbf{D}_\infty(V)$. The following characterization of $\mathbf{D}_\infty(V)$ is due to Goodman (unpublished). For a special case see [19, Corollary 9.3].

THEOREM 1.2. *Let $L \in \mathfrak{U}(\mathfrak{g})$ be elliptic and let $A = \overline{v(L)}$. Then*

$$\mathbf{D}_\infty(V) = \bigcap_{n=0}^{\infty} \mathbf{D}_{A^n}.$$

Furthermore, the topology on $\mathbf{D}_\infty(V)$ defined by the seminorms $x \to \| A^n x \|$, $n = 0, 1, 2, \ldots$ is identical with the topology defined by the family $\{\rho_n\}$.

Proof. Again the nontrivial inclusion follows from the elliptic regularity theorem (see also the proof of Theorem 1.1 in [11]). To prove the last part we note that each seminorm $\| \cdot \|_n$ is continuous in the topology defined by the family $\{\rho_n\}$, and since $\mathbf{D}_\infty$ is a Fréchet space in both topologies they must coincide (the closed graph theorem). Q.E.D.

PROPOSITION 1.2. *For each $x \in \mathbf{D}_\infty$, the mapping $g \to V_\infty(g)x$ is C^∞ from G to $\mathbf{D}_\infty$.*

Proof. Let $x \in \mathbf{D}_\infty$ and let $L = X_1^{\alpha_1} \cdots X_d^{\alpha_d} \in \mathfrak{U}(\mathfrak{g})$ where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ is a set of nonnegative integers. It suffices to show that the mapping $g \to v(L)\, V(g)x$ from G to $\mathbf{B}$ is C^∞ in a neighborhood of the identity e in G. Let $g(t) \to t$ denote the analytic coordinate system used in the proof of Proposition 1.1. Then the mapping $(s, t) \to V(g(s) \cdot g(t))x$ is C^∞ from $\mathbb{R}^{2d}$ to $\mathbf{B}$, but since

$$v(L)\, V(g(t))x = \left(\frac{\partial}{\partial s_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial s_d}\right)^{\alpha_d} V(g(s)\, g(t))x \,\Big|_{s=0}$$

this completes the proof. Q.E.D.

It is clear that the topology on $\mathbf{D}_\infty$ depends only on the behavior of V on the connected component G_0 of the identity of G. This is reflected in the following result.

THEOREM 1.3. *Let* $\mathbf{D}$ *be a dense linear subspace of* $\mathbf{B}$ *which is contained in* $\mathbf{D}_\infty$ *and invariant under the* $V(g)$, $g \in G_0$. *Then* $\mathbf{D}$ *is dense in* $\mathbf{D}_\infty$ *(in the* $\mathbf{D}_\infty$*-topology).*

Proof. Since each $V_\infty(g)$ is continuous on $\mathbf{D}_\infty$ we can assume that $\mathbf{D}$ is a closed subspace of $\mathbf{D}_\infty$, and, hence, that $\mathbf{D}$ is complete in the $\mathbf{D}_\infty$-topology. Then for $x \in \mathbf{D}$ and $\phi \in C_0^\infty(G_0)$ we have $V(\phi)x \in \mathbf{D}$ [4, Chapter III, Section 3], and we want to show that each vector in $\mathbf{D}_\infty$ is a limit of a sequence of vectors of this form.

If $x \in \mathbf{D}_\infty$ there exists a sequence $\{x_n\} \subseteq \mathbf{D}$ such that $x_n \to x$ in $\mathbf{B}$. Then for $L \in \mathfrak{U}(\mathfrak{g})$ we have

$$v(L)\, V(\phi)\, x_n = V(L\phi)\, x_n \to V(L\phi)x = v(L)\, V(\phi)x$$

in $\mathbf{B}$. This means exactly that $V(\phi)\, x_n \to V(\phi)x$ in the $\mathbf{D}_\infty$-topology; so $V(\phi)x \in \mathbf{D}$ for all $\phi \in C_0^\infty(G_0)$. Since $v(L)$ is continuous on $\mathbf{D}_\infty$ we have

$$v(L)\, V(\phi)x = \int \phi(a)\, v(L)\, V(a)x\, da.$$

Therefore, if $\{\phi_n\} \subseteq C_0^\infty(G_0)$ with $\phi_n \geqslant 0$, $\int \phi_n(a)\, da = 1$, and $\operatorname{supp} \phi_n \downarrow \{e\}$ [25], we get

$$v(L)\, V(\phi_n)x \to v(L)\, V(e)x$$

in $\mathbf{B}$ for all $L \in \mathfrak{U}(\mathfrak{g})$. Since we always assume $V(e) = I$ this shows that $V(\phi_n)x \to x$ in $\mathbf{D}_\infty$. Q.E.D.

The following result is a generalization of Theorem 1 in [27].

COROLLARY 1.2. *Let* $\mathbf{D}$ *be a dense subspace of* $\mathbf{B}$ *which is contained in* $\mathbf{D}_\infty$ *and invariant under the* $V(g)$, $g \in G_0$. *Then*

$$\overline{v(L)} = \overline{v(L) \restriction_{\mathbf{D}}}$$

for all $L \in \mathfrak{U}(\mathfrak{g})$.

Proof. Obvious, since $v(L)$ is continuous on $\mathbf{D}_\infty$.

Remark. Theorem 1.3 has a natural analog in case $t \to V(t)$ is a strongly continuous semigroup in a Banach space. A simple modification of the proof gives the following useful result (which is well known in the case of a one-parameter unitary group in a Hilbert space).

COROLLARY 1.3. *Let $t \to V(t)$ be a strongly continuous semigroup in a Banach space* $\mathbf{B}$, *and let A be the infinitesimal generator. Let* $\mathbf{D}$ *be a dense subspace of* $\mathbf{B}$ *which is contained in* $\mathbf{D}_{A^n}$ *for some $n \in \mathbb{N}$ and such that* $V(t)\mathbf{D} \subseteq \mathbf{D}$ *for* $t \in (0, \infty)$. *Then* $\mathbf{D}$ *is a core for* A^n, i.e.,

$$A^n = \overline{A^n |_{\mathbf{D}}}\,.$$

Proof. Since A has a nonvoid resolvent set, A^n is a closed densely defined operator (see, e.g., p. 602 and p. 648 of [8]). Therefore, $\mathbf{D}_{A^n}$ is a Banach space in the graph norm, and it is easily seen that $\mathbf{D}$ is dense in this space. Q.E.D.

Now again let V be a continuous representation of G in a Banach space $\mathbf{B}$.

A vector $x \in \mathbf{B}$ is called an analytic vector for V if the mapping $g \to V(g)x$ is analytic from G to $\mathbf{B}$, or equivalently if the function $g \to \langle V(g)x, x^* \rangle$ is analytic on G for each $x^* \in B^*$. The subspace of analytic vectors is clearly contained in $\mathbf{D}_\infty$, and Nelson [19, Lemma 7.1] showed that a C^∞ vector x is analytic for V iff $\sum_{n=0}^{\infty} s^n/n!(\rho_n(x)) < \infty$ for some $s > 0$.

The following fundamental result is also due to Nelson [19, Theorem 4].

Let $\{X_1, \ldots, X_d\}$ be a basis for $\mathfrak{g}$ and let $\Delta = \sum X_k^2$. Let $(t, a) \to p^t(a)$ be the fundamental solution of the corresponding heat equation on the connected component of the identity G_0 of G. For $t > 0$ let

$$P^t x = \int_{G_0} p^t(a)\, V(a)x\, da \qquad \text{for} \quad x \in \mathbf{B}$$

Then $P^t x$ is an analytic vector for V, and $P^t x \to x$ as $t \to 0$.

It follows from Nelson's investigations (see [19] for details) that $t \to P^t$ is a strongly continuous semigroup of continuous linear operators in $\mathbf{B}$. We have the following result.

THEOREM 1.4. $\overline{v(\Delta)}$ *is the infinitesimal generator of the semigroup* $t \to P^t$.

Proof. Let the infinitesimal generator be denoted by A. For $\phi \in C_0^\infty(G_0)$ and $x \in \mathbf{B}$ we have

$$P^t V(\phi)x = \int_{G_0} (p^t * \phi)(a)\, V(a)x\, da.$$

By the theory of the heat equation in $L^2(G_0)$ we have

$$\frac{\partial}{\partial t}(p^t * \phi)(a) = (p^t * \Delta\phi)(a)$$

for all $a \in G_0$, $t > 0$. Thus (for $h > 0$)

$$h^{-1}[P^h - I]\, P^t V(\phi)x = \frac{1}{h}\int_t^{t+h} P^s V(\Delta\phi)x\, ds.$$

Since the integrand is continuous this gives $P^t V(\phi)x \in \mathbf{D}_A$ and

$$AP^t V(\phi)x = P^t V(\Delta\phi)x.$$

Letting $t \to 0$ we get $V(\phi)x \in \mathbf{D}_A$, and

$$AV(\phi)x = V(\Delta\phi)x = v(\Delta)\, V(\phi)x.$$

Then, by Corollary 1.2 $\overline{v(\Delta)} \subseteq A$.

On the other hand P^t maps $\mathbf{B}$ into the space of analytic vectors, and by Corollary 1.3 this space is a core for A. Hence $A \subseteq \overline{v(\Delta)}$. Q.E.D.

For related results we refer to [15] and [20].

If $\lambda > 0$ is sufficiently large the operator $\lambda I - \overline{v(\Delta)}$ has a bounded inverse. Using Theorem 1.2 we get that each continuous linear functional f on $\mathbf{D}_\infty(V)$ can be represented as a "distribution derivative" of some element $x^* \in B^*$.

COROLLARY 1.4. *Let $f \in \mathbf{D}_\infty(V)^*$. Then there exist $x^* \in \mathbf{B}^*$ and a nonnegative integer n such that*

$$f(x) = \langle(\lambda I - v(\Delta))^n\, x, x^*\rangle$$

for all $x \in \mathbf{D}_\infty(V)$.

2. INTERTWINING BILINEAR FORMS

Let U and V be continuous representations of a Lie group G in Banach spaces $\mathbf{H}$ and $\mathbf{K}$ respectively. The corresponding infinitesimal representations of $\mathfrak{U}(\mathfrak{g})$ on the spaces of C^∞ vectors are denoted by u and v. Suppose $\beta(\cdot, \cdot)$ is a (separately) continuous bilinear form on $\mathbf{D}_\infty(U) \times \mathbf{D}_\infty(V)$ which is group invariant:

$$\beta(U(g)x, V(g)y) = \beta(x, y)$$

for all $(x, y) \in \mathbf{D}_\infty(U) \times \mathbf{D}_\infty(V)$, $g \in G$.

The importance of the study of such intertwining forms was established by Bruhat [5], and in this section we shall study the structure of β from a different point of view. As observed by Bruhat, there exists a unique continuous linear mapping $T : \mathbf{D}_\infty(U) \to \mathbf{D}_\infty(V)^*$ such that $\beta(x, y) = \langle Tx, y\rangle$, and we have $TU_\infty(g) = V_\infty(g^{-1})^* T$ for all $g \in G$. We have the following natural injections

$$\mathbf{D}_\infty(\hat{V}) \subseteq \hat{\mathbf{K}} \subseteq \mathbf{K}^* \subseteq \mathbf{D}_\infty(V)^*,$$

and it is of interest to know when T actually maps $\mathbf{D}_\infty(U)$ into one of these subspaces.

We prove that this is always the case and in fact T maps $\mathbf{D}_\infty(U)$ into $\mathbf{D}_\infty(\hat{V})$ continuously. The method is based on regularity properties of the resolvent of the elliptic operator $v(\Delta)$; a technique which is well known in the theory of partial differential operators.

THEOREM 2.1. *Let U, V, and $\beta(\cdot, \cdot)$ be as before. Then there exists a closed linear operator T from $\mathbf{H}$ to $\hat{\mathbf{K}}$ with $\mathbf{D}_T \supseteq \mathbf{D}_\infty(U)$ and such that*

(1) *T maps $\mathbf{D}_\infty(U)$ into $\mathbf{D}_\infty(\hat{V})$ continuously;*
(2) *$TU(g) = \hat{V}(g)T$ for all $g \in G$;*
(3) *$\beta(x, y) = \langle Tx, y\rangle$ for all $(x, y) \in \mathbf{D}_\infty(U) \times \mathbf{D}_\infty(V)$.*

Proof. Let $\{X_1, \ldots, X_d\}$ be a basis for the Lie algebra $\mathfrak{g}$ of G, and let $\Delta = \sum_{k=1}^{d} X_k^2$. By Theorem 1.4, the operators $\overline{u(\Delta)}$ and $\overline{v(\Delta)}$ are infinitesimal generators of continuous semigroups in $\mathbf{H}$ and $\mathbf{K}$ respectively. Since the spectrum of an infinitesimal generator is contained in a left halfplane we can choose a real number $\lambda > 0$ such that the operator $A = \lambda I - \overline{u(\Delta)}$ has a bounded inverse in $\mathbf{H}$.

This has the advantage that $x \to \| A^n x \|$ is a norm on $\mathbf{D}_\infty(U)$ for $n = 0, 1, 2, \ldots$, and we have

$$\| x \| \leqslant \| A^{-1} \| \cdot \| Ax \| \leqslant \| A^{-1} \|^2 \cdot \| A^2 x \| \leqslant \cdots$$

for all $x \in \mathbf{D}_\infty(U)$. By Theorem 1.2, this family of norms defines the topology on $\mathbf{D}_\infty(U)$. We note that $\mathbf{D}_{A^n}$ is a Banach space in the norm $\| x \|_n = \| A^n x \|$, and by Corollary 1.3 $\mathbf{D}_\infty(U)$ is dense in this space.

We can assume that λ is chosen such that the operator $B = \lambda I - \overline{v(\Delta)}$ has a bounded inverse in $\mathbf{K}$. Then the corresponding statements hold for the family of norms $y \to \| B^n y \|$ on $\mathbf{D}_\infty(V)$.

Now, $\beta(\cdot, \cdot)$ is automatically continuous because we work with Fréchet spaces (see e.g. [24, p. 88]). Hence for some integers m and n

$$| \beta(x, y)| \leqslant \text{const} \| A^m x \| \cdot \| B^n y \|$$

for all $(x, y) \in \mathbf{D}_\infty(U) \times \mathbf{D}_\infty(V)$. By the properties of the norms we can take $m = n$. It follows that the mapping

$$(x, y) \to \beta(A^{-n}x, B^{-n}y)$$

has a unique extension to a continuous bilinear form on $\mathbf{H} \times \mathbf{K}$. In other words, there exists a continuous linear mapping S from $\mathbf{H}$ into $\mathbf{K}^*$ such that

$$\beta(x, y) = \langle SA^n x, B^n y\rangle$$

for all $(x, y) \in \mathbf{D}_\infty(U) \times \mathbf{D}_\infty(V)$.

Let $(x, y) \in \mathbf{D}_\infty(U) \times \mathbf{D}_\infty(V)$ and $X \in \mathfrak{g}$. By Proposition 1.2 the mapping

$$t \to \beta(U(\exp(tX))x, y) = \beta(x, V(\exp(-tX))\,y)$$

is a C^∞ function on $\mathbb{R}$. By differentiation and use of the invariance of $\beta(\cdot, \cdot)$ we get

$$\beta(A^m x, y) = \beta(x, B^m y) \qquad \text{for all} \quad m \in \mathbb{N}.$$

For $m = n$ this gives

$$\langle SA^{2n}x, B^n y\rangle = \langle SA^n x, B^{2n} y\rangle.$$

Since A maps $\mathbf{D}_\infty(U)$ onto itself and B maps $\mathbf{D}_\infty(V)$ onto itself we get

$$\langle SA^n x, y\rangle = \langle Sx, B^n y\rangle$$

for all $(x, y) \in \mathbf{D}_\infty(U) \times \mathbf{D}_\infty(V)$. Then, for $x \in \mathbf{D}_\infty(U)$ it follows that $Sx \in \mathbf{D}_{(B^n)^*}$ and $(B^n)^*Sx = SA^n x$. On the other hand, it is easily seen that $(B^*)^m = (B^m)^*$ for all $m \in \mathbb{N}$, so $SA^n x = (B^*)^n Sx$. Since A leaves $\mathbf{D}_\infty(U)$ invariant, the left side of this equation is an element of $\mathbf{D}_{(B^*)^n}$; hence $Sx \in \mathbf{D}_{(B^*)^{2n}}$. Repeating the argument we find that S maps $\mathbf{D}_\infty(U)$ into $\mathbf{D}_\infty(\hat{V}) = \bigcap_{n=1}^\infty \mathbf{D}_{(B^*)^n}$ (cf. Theorem 1.1). Thus, $SA^n x = \hat{v}(\lambda I - \Delta)^n Sx$ for all $x \in \mathbf{D}_\infty(U)$, and it follows that S maps $\mathbf{D}_\infty(U)$ into $\mathbf{D}_\infty(\hat{V})$ continuously. Now let

$$T_0 = SA^{2n}\,|_{\mathbf{D}_\infty(U)}\,.$$

Since S is continuous the operator $(B^*)^{2n}S$ is closed, and $T_0 \subseteq (B^*)^{2n}S$. Therefore, T_0 has a closure T, and it is easily checked that T has the desired properties. Q.E.D.

Using the notation introduced in the proof of Theorem 2.1 we have

THEOREM 2.2. *The operator T of Theorem 2.1 is unique, and it has the form* $T = (\overline{SA^{2n}})$.

Proof. Suppose T is a closed linear mapping from $\mathbf{H}$ into $\hat{\mathbf{K}}$ with $\mathbf{D}_T \supseteq \mathbf{D}_\infty(U)$ and such that

(1) $TU(g) = \hat{V}(g)T$ for all $g \in G$;
(2) $\beta(x, y) = \langle Tx, y\rangle$ for $(x, y) \in \mathbf{D}_\infty(U) \times \mathbf{D}_\infty(V)$.

Then, clearly $Tx = T_0x$ for all $x \in \mathbf{D}_\infty(U)$. Since a closed operator commutes with integration, we have $TU(\phi) \supseteq \hat{V}(\phi)T$ for all $\phi \in C_0^\infty(G)$ (In fact, if $x \in \mathbf{D}_T$ and $z \in \mathbf{D}_{T^*}$ we have

$$\langle T^*z, U(\phi)x\rangle = \int \phi(a)\langle T^*z, U(a)x\rangle\, da = \langle z, \hat{V}(\phi)\, Tx\rangle.$$

Since T is the adjoint of T^* [24, p. 156], we get $U(\phi)x \in \mathbf{D}_T$ and $TU(\phi)x = \hat{V}(\phi)\, Tx$.) If $x \in \mathbf{D}_T$ and $\{\phi_n\}$ is an approximate identity we have $U(\phi_n)x \to x$ and $T_0U(\phi_n)x = \hat{V}(\phi_n)\, Tx \to Tx$. Hence

$$T \subseteq \bar{T}_0 \subseteq (\overline{SA^{2n}}).$$

On the other hand, let $x \in \mathbf{D}_{A^{2n}}$. By Corollary 1.3, there exists a sequence $\{x_k\}$ in $\mathbf{D}_\infty(U)$ such that $x_k \to x$ and $A^{2n}x_k \to A^{2n}x$. Then $SA^{2n}x_k = T_0x_k \to SA^{2n}x$; so $x \in \mathbf{D}_T$ and $SA^{2n}x = Tx$. Q.E.D.

Remark. If G is connected it is easily seen that $\beta(\cdot, \cdot)$ is group invariant if and only if

$$\beta(u(X)x, y) + \beta(x, v(X)y) = 0$$

for all $X \in \mathfrak{g}$, $(x, y) \in \mathbf{D}_\infty(U) \times \mathbf{D}_\infty(V)$.

Suppose $\mathbf{K}$ is a Hilbert space. Then it is natural to let $\hat{V}(g) = V(g^{-1})^*$, where $*$ now denotes the Hilbert space adjoint relative to the scalar product $\langle\cdot, \cdot\rangle$ on $\mathbf{K}$. In this case it is more convenient to consider sesquilinear forms, and we get the following result.

COROLLARY 2.1. *Let U be as before and let V be a continuous representation in a Hilbert space $\mathbf{K}$. Suppose $\beta(\cdot, \cdot)$ is a continuous group invariant sesquilinear form on $\mathbf{D}_\infty(U) \times \mathbf{D}_\infty(V)$. Then there exists a unique closed linear mapping T from $\mathbf{H}$ to $\mathbf{K}$ such that*

(1) *$\mathbf{D}_T \supseteq \mathbf{D}_\infty(U)$ and T maps $\mathbf{D}_\infty(U)$ into $\mathbf{D}_\infty(\hat{V})$ continuously*;
(2) *$\beta(x, y) = \langle Tx, y\rangle$ for all $(x, y) \in \mathbf{D}_\infty(U) \times \mathbf{D}_\infty(V)$*;
(3) *$TU(g) = \hat{V}(g)T$ for all $g \in G$.*

Remark. If V is unitary (or normal), we have $\mathbf{D}_\infty(\hat{V}) = \mathbf{D}_\infty(V)$, but in general this is not the case. In fact, simple examples show that we can have $\mathbf{D}_\infty(V) \cap \mathbf{D}_\infty(\hat{V}) = \{0\}$.

THEOREM 2.3. *Let U be a continuous unitary representation of G in a Hilbert space* $\mathbf{H}$, *and let $\beta(\cdot, \cdot)$ be a continuous Hermitian sesquilinear form on $\mathbf{D}_\infty(U)$ which is group invariant*:

$$\beta(U(g)x, U(g)y) = \beta(x, y) \qquad \text{for all} \quad g \in G, \quad x, y \in \mathbf{D}_\infty .$$

Then there exists a unique closed symmetric operator T in $\mathbf{H}$ *such that*

$$\beta(x, y) = \langle Tx, y\rangle \qquad \text{for all} \quad x, y \in \mathbf{D}_\infty .$$

This operator has the following properties:

(1) *T is self adjoint and $\mathbf{D}_\infty$ is a core for T.*

(2) *T leaves $\mathbf{D}_\infty$ invariant, and the restriction is a continuous linear operator on $\mathbf{D}_\infty$.*

(3) *$TU(g) = U(g)T$ for all $g \in G$.*

Proof. Using the notation introduced in the proof of Theorem 2.1 we have: $A = \lambda I - \overline{u(\Delta)}$ is a self adjoint operator in $\mathbf{H}$ and $A \geqslant \lambda I$ [20]. Since $\beta(\cdot, \cdot)$ is Hermitian, it follows from the construction of S that S is a bounded self-adjoint operator. Since $SA^{2n} \subseteq A^{2n}S$ it follows that SA^{2n} is essentially self adjoint. Therefore, we must have

$$\overline{(SA^{2n})} = \bar{T}_0 = A^{2n}S;$$

so the operator $\bar{T}_0$ has the desired properties. If T is any closed symmetric operator such that $\beta(x, y) = \langle Tx, y\rangle$ for $x, y \in \mathbf{D}_\infty$, we have $T_0 \subseteq T$ and hence $\bar{T}_0 = T$. Q.E.D.

COROLLARY 2.2. *Let U be a continuous unitary representation of G in a Hilbert space* $\mathbf{H}$, *and suppose T is a symmetric operator in* $\mathbf{H}$ *such that*

(1) $\mathbf{D}_T \supseteq \mathbf{D}_\infty$;

(2) *$TU(g) \supseteq U(g)T$ for all $g \in G$.*

Then T is essentially self-adjoint.

Proof. By the closed graph theorem T maps $\mathbf{D}_\infty(U)$ into $\mathbf{H}$ continuously; so it suffices to take $\beta(x, y) = \langle Tx, y\rangle$ for $x, y \in \mathbf{D}_\infty$. Then $\bar{T}$ has all the properties stated in Theorem 2.3. Q.E.D.

The following result is a generalization of Corollary 2.4 in [20].

COROLLARY 2.3. *Suppose T is a symmetric operator on $\mathbf{D}_\infty(U)$ such that $Tu(L) = u(L)T$ for some elliptic element $L \in \mathfrak{U}(\mathfrak{g})$. Then T is essentially self adjoint.*

Proof. The operator $A = \overline{u(L^*L)}$ is self adjoint [20], and we let $V(t) = e^{itA}$ for $t \in \mathbb{R}$. By Theorem 1.2, $\mathbf{D}_\infty(V) = \mathbf{D}_\infty(U)$ as topological vector spaces, and since T leaves $\mathbf{D}_\infty(U)$ invariant T is automatically continuous on this space. Then $\beta(x, y) = \langle Tx, y \rangle$ is a continuous Hermitian form on $\mathbf{D}_\infty(V)$, and

$$\beta(Ax, y) = \beta(x, Ay) \qquad \text{for all} \quad x, y \in \mathbf{D}_\infty(V).$$

Therefore, β is also invariant under the $V(t)$, $t \in \mathbb{R}$, so the result follows from Corollary 2.2. Q.E.D.

A linear operator T on $\mathbf{D}_\infty(U)$ is called formally normal on $\mathbf{D}_\infty$ if there exists a linear operator T' on $\mathbf{D}_\infty$ such that

$$T' \subseteq T^* \qquad \text{and} \qquad TT' = T'T.$$

COROLLARY 2.4. *Let T be formally normal on $\mathbf{D}_\infty(U)$ and suppose either*

(1) $TU(g) = U(g)T$ *for* $g \in G$; *or*
(2) $Tu(L) = u(L)T$ *for some symmetric elliptic element* $L = L^* \in \mathfrak{U}(\mathfrak{g})$.

Then T is essentially normal.

Proof. We have $T = A + iB$ and $T' = A - iB$, where A and B are symmetric operators on $\mathbf{D}_\infty$ such that $AB = BA$. Since

$$(A^2 + B^2)\, u(L) = u(L)(A^2 + B^2)$$

the operator $A^2 + B^2$ is essentially self adjoint by Corollary 2.3. Then, by Nelson's general criterion [19, Corollary 9.2], the operator $T_1 = \bar{A} + i\bar{B}$ is normal. Since T_1 commutes with the unitary group generated by $\overline{iu(L)}$, we get $T_1 = \bar{T}$ by previous arguments. Q.E.D.

Remark. The normality of T_1 can also be established by more direct arguments. In fact, since A and B commute with $u(L)$ they are essentially self adjoint, and the unitary group $t \to U(t)$ generated by $i\bar{A}$ commutes with the unitary group generated by $\overline{iu(L)}$. Then each $U(t)$, $t \in \mathbb{R}$ leaves $\mathbf{D}_\infty$ invariant, and for each $x \in \mathbf{D}_\infty$ the mapping $t \to U(t)x$ is differentiable from $\mathbb{R}$ to $\mathbf{D}_\infty$. Since B is continuous on $\mathbf{D}_\infty$ the function $f : t \to U(t)\, BU(-t)x$ is differentiable, and

$$f'(t) = U(t)[iA, B]\, U(-t)x = 0 \qquad \text{for all} \quad t \in \mathbb{R}.$$

Therefore, $U(t)\, BU(-t)x = Bx$ for all $t \in \mathbb{R}$, $x \in \mathbf{D}_\infty$, and we get

$$U(t)\bar{B} \subseteq \bar{B}U(t) \qquad \text{for all} \quad t \in \mathbb{R}.$$

It follows that $\bar{A}$ and $\bar{B}$ are commuting self-adjoint operators. Q.E.D.

If $Z \in \mathfrak{U}(\mathfrak{g})$ is a central element, then the operator $T = u(Z)$ is formally normal on $\mathbf{D}_\infty$. In this case the normality of $\overline{u(Z)}$ has been established by Segal [27, Theorem 2].

We close this section with a simple example which shows the rather special nature of the spaces of C^∞ vectors for unitary representations.

EXAMPLE 2.1. Let $\mathscr{S}$ denote the Schwartz space of rapidly decreasing C^∞ functions on $\mathbb{R}$. Let $\lambda \in \mathbb{R}$ and define

$$\begin{aligned} (U(t)f)(x) &= f(x+t) \\ (V(t)f)(x) &= e^{it\lambda}f(x). \end{aligned} \qquad \text{for} \quad f \in \mathscr{S}, \quad t \in \mathbb{R}$$

Then $t \to U(t)$ and $t \to V(t)$ are infinitely differentiable representations of $\mathbb{R}$ in the Fréchet space $\mathscr{S}$, and they form restrictions of continuous unitary representations in $L^2(\mathbb{R})$. Let $\beta(\cdot, \cdot)$ be a continuous sesquilinear form on $\mathscr{S}$ such that

$$\beta(U(t)f, V(t)g) = \beta(f, g) \qquad \text{for} \quad t \in \mathbb{R}, \quad f, g \in \mathscr{S}.$$

On the basis of Corollary 2.1 one would maybe expect that $\beta(\cdot, \cdot)$ has the form

$$\beta(f, g) = \langle Tf, g \rangle,$$

where T is a closed intertwining operator in $L^2(\mathbb{R})$ which maps $\mathscr{S}$ into $\mathscr{S}$ continuously. In order to see that this is not the case we can take

$$\beta(f, g) = \hat{f}(\lambda)\, g(0) \qquad \text{for} \quad f, g \in \mathscr{S},$$

where $\hat{f}$ denotes the Fourier transform of f. Then $\beta(\cdot, \cdot)$ is continuous and group invariant, but there is no linear operator T in $L^2(\mathbb{R})$ such that $\beta(f, g) = \langle Tf, g \rangle$.

It is easy to construct intertwining operators for U and V. For example, let $k \in \mathscr{S}$ and take $Tf = \hat{f}(\lambda) \cdot k$ for $f \in \mathscr{S}$. Then T maps $\mathscr{S}$ into $\mathscr{S}$ continuously and $TU(t) = V(t)T$, but T is not closable in $L^2(\mathbb{R})$. In fact, U and V have no nonzero closed densely defined intertwining operators in $L^2(\mathbb{R})$.

3. Irreducibility and Equivalence

Let U and V be continuous representations of a Lie group G in Banach spaces $\mathbf{H}$ and $\mathbf{K}$ respectively. We recall some general definitions.

Definition 3.1. A representation U is called topologically irreducible if there is no closed linear subspace of $\mathbf{H}$ which is invariant under the $U(g)$, $g \in G$, other than $\{0\}$ and $\mathbf{H}$.

Definition 3.2. A representation U is called operator irreducible in case it has the following property: If T is a closed densely defined linear operator in $\mathbf{H}$ such that

$$TU(g) = U(g)T \qquad \text{for all} \quad g \in G,$$

then $T = \lambda I$ for some $\lambda \in \mathbb{C}$.

Definition 3.3. U and V are called weakly equivalent if there exists a closed densely defined injective operator T with a dense range such that

$$TU(g) = V(g)T \qquad \text{for all} \quad g \in G.$$

If there is a continuous isomorphism T of $\mathbf{H}$ onto $\mathbf{K}$ with this property, U and V are called equivalent.

Remark. The notion of weak equivalence of representations was introduced by Naimark in connection with this work on the Lorentz group (see [18, 29] for references). Further investigations can be found in the work of Želobenko on complex semisimple groups [30, 31]. A general discussion of "Naimark related" pairs of representations was given by Fell [9]. This paper also contains some interesting examples.

The notion of operator irreducibility adopted in the present paper is a special case of a definition proposed by Gelfand, Graev and Vilenkin in their book which also contains some examples [10].

We remark that some of the results of the present section are also valid for representations in more general spaces, but here we are mainly concerned with Hilbert space representations.

Before the discussion of equivalence and irreducibility we establish some general properties of closed intertwining operators. Suppose T

is a closed linear operator from $\mathbf{H}$ to $\mathbf{K}$. We let T_∞ denote the restriction of T to the following domain

$$\mathbf{D}_{T_\infty} = \{x \in \mathbf{D}_\infty(U) \mid x \in \mathbf{D}_T \text{ and } Tx \in \mathbf{D}_\infty(V)\}.$$

Then T_∞ is a closed linear operator from $\mathbf{D}_\infty(U)$ to $\mathbf{D}_\infty(V)$.

THEOREM 3.1. *Let T be a closed (densely defined) linear mapping from $\mathbf{H}$ to $\mathbf{K}$ such that*

$$TU(g) = V(g)T \qquad \text{for all} \quad g \in G.$$

Then, T_∞ is a closed (densely-defined) linear mapping from $\mathbf{D}_\infty(U)$ to $\mathbf{D}_\infty(V)$ and

$$T_\infty U_\infty(g) = V_\infty(g)\, T_\infty \qquad \text{for all} \quad g \in G.$$

If, furthermore, $\mathbf{D}_T \supseteq \mathbf{D}_\infty(U)$, then T_∞ is a continuous linear mapping of $\mathbf{D}_\infty(U)$ into $\mathbf{D}_\infty(V)$.

Proof. Since a closed linear operator commutes with integration we have

$$TU(\phi) \supseteq V(\phi)T \qquad \text{for all} \quad \phi \in C_0^\infty(G).$$

Thus, $U(\phi)x \in \mathbf{D}_{T_\infty}$ for all $x \in \mathbf{D}_T$, $\phi \in C_0^\infty(G)$. By Theorem 1.3 T_∞ is densely defined in $\mathbf{D}_\infty(U)$ if and only if T is densely defined in $\mathbf{H}$. Also T_∞ has a dense range in $\mathbf{D}_\infty(V)$ if and only if T has a dense range in $\mathbf{K}$. Suppose $\mathbf{D}_T \supseteq \mathbf{D}_\infty(U)$. By the closed graph theorem T maps $\mathbf{D}_\infty(U)$ into $\mathbf{K}$ continuously. Then for $x \in \mathbf{D}_\infty(U)$, the mapping $g \to TU_\infty(g)x$ is C^∞ from G to $\mathbf{K}$ by Proposition 1.2. Since T intertwines the representations, this shows that Tx is a C^∞ vector for V; so T_∞ is defined on all of $\mathbf{D}_\infty(U)$. Q.E.D.

THEOREM 3.2. *Let S be a closed linear operator from $\mathbf{D}_\infty(U)$ to $\mathbf{D}_\infty(V)$ such that*

$$SU_\infty(g) = V_\infty(g)S \qquad \text{for all} \quad g \in G.$$

Then S has a closed extension $\bar{S}$ from $\mathbf{H}$ to $\mathbf{K}$ which intertwines U and V.

Proof. In order to see that the closure in $\mathbf{H} \times \mathbf{K}$ of the graph of S is the graph of a linear operator we assume

$$\{x_n\} \subseteq \mathbf{D}_S, \qquad x_n \to 0 \text{ in } \mathbf{H} \qquad \text{and} \qquad Sx_n \to y \text{ in } \mathbf{K}.$$

Then we have to show $y = 0$. Since $U(\phi)$ is a continuous linear mapping from $\mathbf{H}$ into $\mathbf{D}_\infty(U)$ for each $\phi \in C_0^\infty(G)$, we get $U(\phi)\, x_n \to 0$ in $\mathbf{D}_\infty(U)$ and $SU_\infty(\phi)\, x_n = V_\infty(\phi)\, Sx_n \to V(\phi)\, y$ in $\mathbf{D}_\infty(V)$. Since S is closed, we have $V(\phi)\, y = 0$ for all $\phi \in C_0^\infty(G)$; hence $y = 0$. Q.E.D.

Remark. It was seen in Example 2.1 that the previous result does not necessarily hold if $\mathbf{D}_\infty(U)$ and $\mathbf{D}_\infty(V)$ are replaced by some other nice Fréchet spaces. However, the smoothing arguments used here are also valid in some of the other "natural topological spaces" of differentiable vectors.

COROLLARY 3.1. *U and V are weakly equivalent if and only if U_∞ and V_∞ are weakly equivalent.*

Proof. We have already established the bijective correspondence $T \to T_\infty$ between closed intertwining operators. A simple smoothing argument shows T is injective if and only if T_∞ is injective. Q.E.D.

Similar arguments give the following result, the first part of which is due to Bruhat [5, Proposition 2.6].

COROLLARY 3.2.

(1) *U is topologically irreducible if and only if U_∞ is topologically irreducible.*

(2) *U is operator irreducible if and only if U_∞ is operator irreducible.*

THEOREM 3.3. *Let T be a continuous linear mapping of $\mathbf{D}_\infty(U)$ into $\mathbf{D}_\infty(V)$. Then*

$$Tu(X) = v(X)T \qquad \textit{for all} \quad X \in \mathfrak{g}$$

if and only if

$$TU(g) = V(g)T \qquad \textit{for all} \quad g \in G_0$$

(*as usual G_0 denotes the connected component of the identity of G.*)

Proof. Suppose $Tu(X) = v(X)T$ for $X \in \mathfrak{g}$. Let $x \in \mathbf{D}_\infty(U)$ and define

$$F(t) = V(\exp(tX))\, TU(\exp(-tX))x \qquad \text{for} \quad t \in \mathbb{R}.$$

By Proposition 1.2 F is a differentiable mapping from $\mathbb{R}$ into $\mathbf{D}_\infty(V)$, and we have

$$\begin{aligned} F'(t) &= V(\exp(tX))[v(X)T - Tu(X)]\, U(\exp(-tX))x \\ &= 0 \qquad \text{for all} \quad t \in \mathbb{R}. \end{aligned}$$

Therefore F must be constant, so

$$V(\exp(tX))\, TU(\exp(-tX))x = Tx \qquad \text{for} \quad t \in \mathbb{R}.$$

Using the coordinate system $g(t) \to t$ from the proof of Proposition 1.1 we get $V(g)T = TU(g)$ for all g in a neighborhood of e in G. Since G_0 is generated by any such neighborhood this relation holds for all g in G_0. The other implication is immediate by Proposition 1.2.

Q.E.D.

COROLLARY 3.3. *If G is connected the following statements are equivalent*

(1) *U_∞ and V_∞ are equivalent*;

(2) *there exists a continuous linear isomorphism T of $\mathbf{D}_\infty(U)$ onto $\mathbf{D}_\infty(V)$ such that*

$$Tu(X) = v(X)T \qquad \textit{for all} \quad X \in \mathfrak{g}.$$

It is well known that the notion of weak equivalence of two representations U and V is unsatisfactory in general, and one could try to restrict it by requiring the intertwining operator T to have $\mathbf{D}_\infty(U)$ in its domain. By Theorem 3.1 this makes the relation transitive, but unfortunately it loses symmetry. (Of course, we could furthermore require T to have $\mathbf{D}_\infty(V)$ in its range. Then U and V would be "equivalent" if and only if U_∞ and V_∞ are equivalent.) In case V is a unitary representation it follows from Corollary 2.1 that U is "equivalent to" V if and only if the representations U_∞ and V_∞ are "form equivalent". The definition seems very reasonable for certain types of representations, but the following example shows that this is not the case in general.

EXAMPLE 3.1. Let G be the group of all 2×2 matrices of the form

$$g = \begin{pmatrix} e^t & s \\ 0 & 1 \end{pmatrix} \qquad s, t \in \mathbb{R}.$$

Take $\mathbf{K} = L^2(0, \infty)$ and let

$$(V(g)f)(x) = e^{isxe^{t/2}} f(e^t x) \qquad \text{for} \quad f \in \mathbf{K}.$$

Then $g \to V(g)$ is a continuous irreducible unitary representation of G in $\mathbf{K}$.

Let $\mathbf{H} = \{f \in L^2(0, \infty) \mid f' \in L^2(0, \infty)\}$ (here f' denotes the distribution derivation of f [23]). Then $\mathbf{H}$ is a Hilbert space in the norm $\|\cdot\|$ defined by

$$\|f\|^2 = \int_0^\infty |f(x)|^2 \, dx + \int_0^\infty |f'(x)|^2 \, dx \qquad \text{for} \quad f \in \mathbf{H}.$$

It is clear that each $V(g)$ leaves $\mathbf{H}$ invariant, and we let $U(g)$ denote the restriction to $\mathbf{H}$ of $V(g)$ for all $g \in G$. Then $g \to U(g)$ is a continuous representation of G in $\mathbf{H}$, and if T denotes the natural injection of $\mathbf{H}$ into $\mathbf{K}$ we have

$$TU(g) = V(g)T \qquad \text{for all} \quad g \in G.$$

In particular, U and V are weakly equivalent, and T maps $\mathbf{D}_\infty(U)$ into $\mathbf{D}_\infty(V)$ continuously. It is easy to see that there is no closed linear operator S ($\neq 0$) from $\mathbf{K}$ to $\mathbf{H}$ such that $\mathbf{D}_S \supseteq \mathbf{D}_\infty(V)$ and $SV(g) = U(g)S$ for all $g \in G$. This shows that the restricted notion of weak equivalence is not symmetric. We remark that the representation U is not topologically irreducible. In fact, $\{f \in \mathbf{H} \mid f(0) = 0\}$ is a closed subspace of $\mathbf{H}$ which is invariant under the $U(g)$, $g \in G$. It seems to be difficult to show that U is operator irreducible, but U does have the following (weaker) property. If T is a continuous linear operator on $\mathbf{H}$ such that $TU(g) = U(g)T$ for all $g \in G$, then $T = \lambda I$ for some $\lambda \in \mathbb{C}$.

THEOREM 3.4. *Suppose U and V are continuous unitary representations of G. Then the following statements are equivalent.*

(1) *U and V are unitarily equivalent;*

(2) *U_∞ and V_∞ are equivalent;*

(3) *There exists a continuous nondegenerate sesquilinear form $\beta(\cdot, \cdot)$ on $\mathbf{D}_\infty(U) \times \mathbf{D}_\infty(V)$ such that $\beta(U(g)x, V(g)y) = \beta(x, y)$ for all $g \in G$, $(x, y) \in \mathbf{D}_\infty(U) \times \mathbf{D}_\infty(V)$.*

(4) *U and V are weakly equivalent.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. Here $\beta(\cdot, \cdot)$ is said to be nondegenerate in case it has the following two properties:

(i) If $\beta(x, y) = 0$ for all $x \in \mathbf{D}_\infty(U)$, then $y = 0$.

(ii) If $\beta(x, y) = 0$ for all $y \in \mathbf{D}_\infty(V)$, then $x = 0$.

If $\beta(\cdot, \cdot)$ is such a form, it follows from Corollary 2.1 that U_∞ and V_∞ are weakly equivalent. By Corollary 3.1 this implies (4). Suppose T defines a weak equivalence between U and V (see Definition 3.3), and let $T = W \mid T \mid$ be the polar decomposition. Since T is injective, W is an isometry, and because T has a dense range W is surjective. In other words, W is a unitary mapping of $\mathbf{H}$ onto $\mathbf{K}$ and we have [29, Proposition 2]

$$WU(g) = V(g)W \qquad \text{for all} \quad g \in G.$$

We note that $\mid T \mid = (T^*T)^{1/2}$ is a self-adjoint operator in $\mathbf{H}$ and $\mid T \mid U(g) = U(g) \mid T \mid$ for all $g \in G$. Q.E.D.

It is well known that a continuous unitary representation U is topologically irreducible if and only if the commutant $U(G)'$ is trivial [6]. Using the polar decomposition this is easily seen to be equivalent to operator irreducibility.

COROLLARY 3.4. *Let U be a continuous unitary representation of G in a Hilbert space* $\mathbf{H}$. *The following statements are equivalent.*

(1) *U is operator irreducible.*

(2) *U_∞ is topologically irreducible.*

(3) *If $\beta(\cdot, \cdot)$ is a continuous group invariant sesquilinear form on $\mathbf{D}_\infty(U)$, then $\beta(x, y) = \text{const.}\langle x, y\rangle$ for all $x, y \in \mathbf{D}_\infty$.*

COROLLARY 3.5. *If G is a connected Lie group the following statements are equivalent.*

(1) *U is irreducible.*

(2) *If T is a continuous linear operator on $\mathbf{D}_\infty(U)$ such that $Tu(X) = u(X)T$ for all $X \in \mathfrak{g}$, then $T = \lambda I$ for some $\lambda \in \mathbb{C}$.*

(3) *If $\beta(\cdot, \cdot)$ is a continuous sesquilinear form on $\mathbf{D}_\infty$ such that $\beta(u(X)x, y) + \beta(x, u(X)y) = 0$ for all $x, y \in D_\infty$, $X \in \mathfrak{g}$, then $\beta(x, y) = \lambda\langle x, y\rangle$ for some $\lambda \in \mathbb{C}$.*

Proof. By Theorem 3.3 (or Proposition 1.2) $\beta(\cdot, \cdot)$ is G-invariant if and only if it is $\mathfrak{g}$-invariant. Q.E.D.

Remark. It is not true in general that the representation u of $\mathfrak{U}(\mathfrak{g})$ is topologically irreducible in $\mathbf{D}_\infty(U)$. For example, let U be the irreducible unitary representation of Example 5.1 and take

$$D = \{f \in \mathcal{S} \mid f^{(n)}(x) = 0 \text{ for } \mid x \mid \leqslant 1, n = 0, 1, 2,\ldots\}.$$

Then D is a closed $\mathfrak{g}$-invariant subspace of $\mathcal{S}$, but D is not dense in $\mathcal{S}$ (or in L^2).

If D contains analytic vectors, this can no longer happen. In fact, let U be a continuous representation of a connected Lie group in a Banach space. Then it is easy to show that U is topologically irreducible if and only if each nonzero analytic vector x for U is cyclic for u in $\mathbf{D}_\infty$ (i.e., $\{u(L)x \mid L \in \mathfrak{U}(\mathfrak{g})\}$ is dense in $\mathbf{D}_\infty$).

4. Some Special Banach Space Representations

In this section we study families of Banach space representations having a common space of C^∞ vectors. If the family contains an irreducible unitary representation, the results in Section 3 give some additional information about irreducibility of all representations in the family.

Theorem 4.1. *Let $g \to V(g)$ be a continuous representation of a Lie group G in a Banach space $\mathbf{B}$. Let $\|\cdot\|'$ be a continuous norm on $\mathbf{D}_\infty(V)$ (i.e., in the $\mathbf{D}_\infty$-topology) and suppose there exists a nonnegative real valued function $c(\cdot)$ on G which is bounded on some neighborhood of $e \in G$ and such that*

$$\| V_\infty(g)x \|' \leqslant c(g) \| x \|' \qquad \text{for all} \quad g \in G, \quad x \in \mathbf{D}_\infty .$$

Let $\mathbf{B}'$ denote the completion of $\mathbf{D}_\infty$ in the norm $\|\cdot\|'$, and let $V'(g)$ be the extension to $\mathbf{B}'$ of $V_\infty(g)$ for $g \in G$. Then $g \to V'(g)$ is a continuous representation of G in $\mathbf{B}'$, and $\mathbf{D}_\infty(V') \supseteq \mathbf{D}_\infty(V)$.

Proof. For each $g \in G$, $V_\infty(g)$ has a unique extension to a continuous linear operator $V'(g)$ on the Banach space $\mathbf{B}'$, and $\| V'(g)\|' \leqslant c(g)$ for $g \in G$. In particular, $\| V'(g)\|' \leqslant$ const for all g in a neighborhood of e in G. It is easily seen that V' has the group property $V'(g_1 g_2) = V'(g_1)\, V'(g_2)$; so in order to prove the continuity it suffices to show that the mapping $g \to V'(g)x$ is continuous at $e \in G$ for each $x \in \mathbf{D}_\infty$.

By the continuity of $\|\cdot\|'$ there exists a $k > 0$ such that (for some $n > 0$)

$$\| x \|' \leqslant k \sum_{m=0}^{n} \rho_m(x) \qquad \text{for all} \quad x \in \mathbf{D}_\infty .$$

Here ρ_m denotes the seminorm determined by V as defined in Section 1. For each $x \in \mathbf{D}_\infty$, $V'(g)x = V_\infty(g)x$; so the continuity is clear from Proposition 1.2. In fact, the mapping $g \to V'(g)x$ is C^∞ from G to $\mathbf{B}'$. Therefore, $\mathbf{D}_\infty(V') \supseteq \mathbf{D}_\infty$, and if v' denotes the infinitesimal representation on $\mathbf{D}_\infty(V')$ it is easily seen that

$$v'(L)x = v(L)x \qquad \text{for} \quad x \in \mathbf{D}_\infty(V) \qquad \text{and} \qquad L \in \mathfrak{U}(\mathfrak{g}).$$

COROLLARY 4.1. *Let V be an irreducible representation, and let $\|\cdot\|'$ be a norm on $\mathbf{D}_\infty$ which satisfies the hypothesis of Theorem* 4.1. *Suppose that the topology on $\mathbf{D}_\infty$ defined by the seminorms $x \to \| v(L)x \|'$, $L \in \mathfrak{U}(\mathfrak{g})$ is equivalent to the original one. Then V' is an irreducible representation in $\mathbf{B}'$ and $\mathbf{D}_\infty(V') = \mathbf{D}_\infty(V)$.*

Proof. The original topology on $\mathbf{D}_\infty$ is defined by the seminorms $x \to \| v(L)x \|$, $L \in \mathfrak{U}(\mathfrak{g})$. In the proof of Theorem 4.1 we noticed that $v(L)x = v'(L)x$ for $x \in \mathbf{D}_\infty$; hence it follows that $\mathbf{D}_\infty$ is a closed subspace of $\mathbf{D}_\infty(V')$. On the other hand, Theorem 1.3 shows that $\mathbf{D}_\infty(V)$ is dense in $\mathbf{D}_\infty(V')$; so they must coincide. By Corollary 3.2, V' is topologically (operator) irreducible if and only if V is topologically (operator) irreducible. Q.E.D.

Now let U be a continuous irreducible unitary representation of G in a Hilbert space $\mathbf{H}$. Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are two (not necessarily distinct) norms on $\mathbf{D}_\infty(U)$ each of which satisfies the conditions of Corollary 4.1, and let V_1 and V_2 denote the corresponding irreducible representations in $\mathbf{B}_1$ and $\mathbf{B}_2$, respectively. With this notation we have the following result.

COROLLARY 4.2.

(a) *The vector space of continuous intertwining sesquilinear forms for V_1 and V_2 is at most one-dimensional.*

(b) *If V_1 and V_2 has a nonzero intertwining operator T, then $\mathbf{B}_1 \subseteq \mathbf{B}_2$ and there exists a $\lambda \neq 0$ such that $Tx = \lambda x$ for all $x \in \mathbf{B}_1$.*

Proof. Let β be an intertwining sesquilinear form for V_1 and V_2. Then the restriction of β to $\mathbf{D}_\infty$ is continuous and it is invariant under the $U_\infty(g)$, $g \in G$. By Corollary 3.4

$$\beta(x, y) = \lambda\langle x, y\rangle \qquad \text{for all} \quad x, y \in \mathbf{D}_\infty .$$

Since β is uniquely determined by its values on $\mathbf{D}_\infty$ this proves (a).

Suppose $T : \mathbf{B}_1 \to \mathbf{B}_2$ is an intertwining operator for V_1 and V_2. $\| Tx \|_2 \leqslant c \| x \|_1$ for all $x \in \mathbf{B}_1$. Then, by Theorem 3.1, T leaves $\mathbf{D}_\infty$ invariant, and the restriction of T to $\mathbf{D}_\infty$ is a continuous linear operator on this space. By Corollary 3.4, there exists a $\lambda \in \mathbb{C}$ such that $Tx = \lambda x$ for $x \in \mathbf{D}_\infty$. Hence $|\lambda| \| x \|_2 \leqslant c \| x \|_1$ on $\mathbf{D}_\infty$, and, if $T \neq 0$, we have $\lambda \neq 0$. Therefore, $\mathbf{B}_1 \subseteq \mathbf{B}_2$ and the inclusion map is continuous. Clearly, $Tx = \lambda x$ for $x \in \mathbf{B}_1$.

Note that if T maps $\mathbf{B}_1$ onto $\mathbf{B}_2$ the spaces coincide and the two norms are equivalent. Q.E.D.

5. Examples

It is known that many of the spaces used in the theory of partial differential operators have a natural connection with group representations. As an illustration of the theory in Section 4 we choose some of these spaces as examples.

Example 5.1. Let G be the Heisenberg group [22], i.e., the group of all real 3×3 matrices of the form

$$g = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

The Lie algebra is generated by elements X, Y, and Z satisfying the commutation relations

$$[X, Y] = Z, \qquad [X, Z] = [Y, Z] = 0.$$

The standard representation (or Schrödinger representation) of G is realized in $L^2(\mathbb{R})$ in the following way

$$(U(g)f)(x) = e^{ic}e^{ibx}f(x + a), \qquad f \in L^2(\mathbb{R}).$$

Using Goodman's theorem (cf. Section 1) it is easily seen that the space of C^∞ vectors for the representation $g \to U(g)$ is exactly the Schwartz space $\mathscr{S}$. On $\mathscr{S}$ the infinitesimal representation u is given by

$$u(X) \equiv iP = \frac{d}{dx}, \qquad u(Y) \equiv iQ = ix, \qquad u(Z) = i1,$$

where we have introduced the conventional operators P and Q, and the topology on $\mathscr{S}$ can be defined by the seminorms

$$\phi \to \| Q^n P^m \phi \|_2, \qquad n, m = 0, 1, 2,... .$$

The following discussion is easily modified to include the case $p = +\infty$, but for simplicity we will assume $1 \leqslant p < \infty$.

It is well known that the L^p-norm $\| \cdot \|_p$ is continuous on $\mathscr{S}$, and using a Sobolev lemma it is easily seen that the topology on $\mathscr{S}$ can be defined by the seminorms

$$\phi \to \| Q^n P^m \phi \|_p, \qquad n, m = 0, 1, 2,... .$$

By Theorem 4.1 and Corollary 4.1 the representation U_∞ in $\mathscr{S}$ has an extension to a continuous (topologically and operator) irreducible representation V_p in $L^p(\mathbb{R})$ and $\mathbf{D}_\infty(V_p) = \mathscr{S}$.

If β is a nonzero intertwining sesquilinear form for V_p and V_q we get from Corollary 4.2 that (for some $\lambda \neq 0$)

$$\beta(\phi, \psi) = \lambda \cdot \int \phi(x)\,\overline{\psi(x)}\,dx \qquad \text{for all} \quad \phi, \psi \in \mathscr{S}.$$

This is possible if and only if $1/p + 1/q = 1$.

If T is a continuous intertwining operator for V_p and V_q it follows from Corollary 4.2 that $T = 0$ or $p = q$ in which case $T = \lambda 1$ for some $\lambda \in C$.

If T is a closed densely defined intertwining operator for V_p and V_q ($1 \leqslant p, q < \infty$), we get $T_\infty\phi = \lambda\phi$ for all $\phi \in \mathscr{S}$. Therefore $T\phi = \lambda\phi$ for all $\phi \in \mathbf{D}_T$. By construction, the representations V_p, $1 \leqslant p < \infty$, are all weakly equivalent, but they are far from being equivalent in the classical sense. The example could be generalized by introducing a tempered weight function in the definition of the L^p-norm. Instead of doing this we consider a slightly different situation.

Let k be a tempered weight function on $\mathbb{R}$ in the sense of Hörmander [13, pp. 34–37], i.e., k is a positive real function and there exist constants C and N such that

$$k(x + y) \leqslant (1 + C \mid x \mid)^N\, k(y) \qquad \text{for} \quad x, y \in \mathbb{R}.$$

For $1 \leqslant p < \infty$ we let $B_{p,k}$ denote the space of all tempered distributions f such that the Fourier transformed $\hat{f}$ is a function and

$$\| f \|_{p,k} = \left\{ \int \mid k(x) \hat{f}(x) \mid^p dx \right\}^{1/p} < \infty.$$

Then $B_{p,k}$ is a Banach space with the norm $\| \cdot \|_{p,k}$, and $\mathscr{S}$ is dense in this space. The norm $\| \cdot \|_{p,k}$ is continuous on $\mathscr{S}$, and in fact the topology on $\mathscr{S}$ can be defined by the seminorms:

$$\phi \to \| Q^n P^m \phi \|_{p,k}, \qquad n, m = 0, 1, 2, \ldots .$$

For $\phi \in \mathscr{S}$ we have $\| U_\infty(g)\phi \|_{p,k} \leqslant (1 + C \mid b \mid)^N \| \phi \|_{p,k}$.

It follows from Theorem 4.1 and Corollary 4.1 that the representation in $\mathscr{S}$ has an extension to a continuous irreducible representation V in $B_{p,k}$ and $\mathbf{D}_\infty(V) = \mathscr{S}$.

Again, one can compare the different representations. Properties of the spaces $B_{p,k}$ can be found in [13].

The same results hold for all the different irreducible representations of G [22], and they remain valid for any (finite) number of degrees of freedom. For other examples of a similar nature see [7].

EXAMPLE 5.2. Let G be a Lie group and let $1 \leqslant p < \infty$. We form $L^p(G)$ for some right invariant Haar measure dx on G. The regular representations U_p and V_p of G in $L^p(G)$ are defined by

$$(U_p(g)f)(x) = f(g^{-1}x); \qquad (V_p(g)f)(x) = f(x \cdot g)$$

for $g \in G$, $f \in L^p(G)$. Each $X \in \mathfrak{g}$ is identified with the right invariant differential operator

$$(Xf)(x) = \frac{d}{dt} f(\exp(-tX) \cdot x) \Big|_{t=0},$$

and the corresponding left invariant differential operator $\tilde{X}$ is defined by

$$(\tilde{X}f)(x) = \frac{d}{dt} f(x \cdot \exp(tX)) \Big|_{t=0}$$

for all differentiable functions f on G. Le $\{X_1, \ldots, X_d\}$ be a basis for $\mathfrak{g}$. We use the multi-index notation $X^\alpha = X_1^{\alpha_1} \cdots X_d^{\alpha_d}$ ($\tilde{X}^\alpha = \tilde{X}_1^{\alpha_1} \cdots \tilde{X}_d^{\alpha_d}$ respectively) if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ is a set of nonnegative integers. As usual $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$. We let δ denote the modular function on G. Specifically, δ is defined by the equation

$$\int_G \phi(g \cdot x)\, dx = \delta(g)^{-1} \int_G \phi(x)\, dx$$

for all $g \in G$, $\phi \in C_0^\infty(G)$. We have the following Sobolev lemma.

LEMMA 5.1. *Let $1 \leqslant p < \infty$ and let s be an integer with $s > d/p$. For each compact neighborhood K of the identity e of G there exists a constant C such that*

$$|f(e)| \leqslant C \sum_{|\alpha| \leqslant s} \left\{ \int_K |\tilde{X}^\alpha f(x)|^p \, dx \right\}^{1/p}$$

for all differentiable functions f on G.

Proof. If $\Omega \subseteq \mathbb{R}^d$ is a suitably nice domain, the Sobolev space $L^{p,s}(\Omega)$ contains only continuous functions by Sobolev's theorem (see e.g. [8, p. 1686]). A simple application of the closed graph theorem shows that point evaluation is a continuous linear functional on $L^{p,s}(\Omega)$, and using local coordinates we get the desired inequality. Q.E.D.

Remark. For $p = 1$ (and hence for all p) we can take $s = d$ in Lemma 5.1 (see [21] for a simple proof which does not make use of Sobolev's theorem).

By Theorem 1.3, $C_0^\infty(G)$ is dense in $\mathbf{D}_\infty(U_p)$, and using Lemma 5.1 it is easy to prove the following results. (The proof is similar to the proof of Theorem 5.1 (below); so we omit the details.) We have

$$\mathbf{D}_\infty(U_p) = \{f \in C^\infty(G) \mid X^\alpha f \in L^p(G) \text{ for all } \alpha\},$$

and again the spaces are well known. (For the case $G = \mathbb{R}^d$ we refer to Schwartz [23, p. 199] for details.) For all $f \in \mathbf{D}_\infty(U_p)$ we have $\|f\|_\infty \leqslant C' \sum_{|\alpha| \leqslant s} \| X^\alpha f \|_p$ and $X^\alpha f$ vanishes at infinity for all α. For $q \geqslant p$ $\| \cdot \|_q$ is a continuous norm on $\mathbf{D}_\infty(U_p)$, and the representation given in Theorem 4.1 is just U_q. We have $\mathbf{D}_\infty(U_p) \subseteq \mathbf{D}_\infty(U_q)$, and the inclusion mapping is continuous.

Each continuous linear functional F on $\mathbf{D}_\infty(U_p)$ is a distribution of finite order. In fact, by Corollary 1.4, F has the form $(1 - \Delta)^n h$ where $h \in L^{p'}$ $(1/p + 1/p' = 1)$. Suppose T is a closed linear mapping from $L^p(G)$ to $L^q(G)$ such that $\mathbf{D}_T \supseteq \mathbf{D}_\infty(U_p)$ and $TU_p(g) = U_q(g)T$ for $g \in G$. By Theorem 3.1, T maps $\mathbf{D}_\infty(U_p)$ into $\mathbf{D}_\infty(U_q)$ continuously, and $\mathbf{D}_\infty(U_p)$ is a core for T. Let $F \in \mathbf{D}_\infty(U_p)^*$ such that $(Tf)(e) = \langle f, F \rangle$ for all $f \in \mathbf{D}_\infty(U_p)$. Using the traditional notation we have

$$(Tf)(g) = \int f(g \cdot x)\, dF(x) \qquad \text{for all} \quad g \in G,$$

so T is a "convolution operator" (see also [1] or [23, p. 197]). Similarly, we get $\mathbf{D}_\infty(V_p) = \{ f \in C^\infty(G) \mid \tilde{X}^\alpha f \in L^p(G) \text{ for all } \alpha)$ and $\| \delta^{1/p} \cdot f \|_\infty \leqslant C \sum_{|\alpha| \leqslant s} \| \tilde{X}^\alpha f \|_p$ for all $f \in \mathbf{D}_\infty(V_p)$. We do not know whether such functions are necessarily bounded on G, but $\delta^{1/p} \cdot (\tilde{X}^\alpha f)$ vanishes at infinity for all α. Finally we remark that each analytic vector f for V_p (or U_p) is an analytic function on G (cf. Corollary 5.2).

As our final example we characterize the space of C^∞ vectors of an induced representation. Since we use some definitions and results from Blattner's paper, it is convenient to follow Blattner's notation [3].

EXAMPLE 5.3. Let G be a Lie group and let H be a closed subgroup. We choose some right invariant Haar measures on G and H and we let Δ and δ denote the respective modular functions. Let

$M = G/H$ denote the right coset space and let π be the projection of G onto M.

Suppose L is a continuous unitary representation of H in a Hilbert space $\mathbf{V}$, and let F^* be the set of functions f from G to $\mathbf{V}$ satisfying the following conditions:

(i) $f(\cdot)$ is measurable.
(ii) $f(\xi \cdot x) = \Delta(\xi)^{-1/2}\, \delta(\xi)^{1/2} L(\xi) f(x)$ for $\xi \in H$ and $x \in G$.
(iii) $\|f(\cdot)\|^2$ is locally integrable.

Each such function f defines a Radon measure μ_f on M via the equation

$$\int_G \|f(x)\|^2 \phi(x)\, dx = \int_M (\tau\phi)(p)\, d\mu_f(p)$$

where $\phi \in C_0(G)$ and $(\tau\phi)(\pi(x)) = \int_H \phi(\xi \cdot x)\, d\xi$. We set $\|f\| = \mu_f(M)^{1/2}$ and $F = \{f \in F^* \mid \|f\| < \infty\}$. If we identify functions in F which are equal locally almost everywhere (l.a.e.) we get a Hilbert space $\mathbf{H}^L$, and the representation U^L of G is defined in the following way:

$$(U^L(g)f)(x) = f(x \cdot g) \qquad \text{for} \quad f \in \mathbf{H}^L.$$

Let $C^\infty(G, \mathbf{V})$ denote the space of infinitely differentiable functions from G to $\mathbf{V}$. Then we have the following result:

THEOREM 5.1.

$$\mathbf{D}_\infty(U^L) = \{f \in C^\infty(G, \mathbf{V}) \mid \tilde{X}^\alpha f \in \mathbf{H}^L \textit{ for all } \alpha\}.$$

Proof. Let $f \in C^\infty(G, \mathbf{V})$ and $X \in \mathfrak{g}$ and suppose $f \in \mathbf{H}^L$ and $\tilde{X}f \in \mathbf{H}^L$. Then for $t \neq 0$

$$\frac{1}{t}[f(y \cdot \exp(tX)) - f(y)] = \frac{1}{t}\int_0^t \tilde{X}f(y \cdot \exp(sX))\, ds.$$

Hence, by Hölder's inequality (suppose $t > 0$):

$$\left\| \frac{1}{t}[f(y \cdot \exp(tX)) - f(y)] - \tilde{X}f(y) \right\|^2$$
$$\leqslant \frac{1}{t}\int_0^t \|\tilde{X}f(y \cdot \exp(sX)) - \tilde{X}f(y)\|^2\, ds.$$

If $\phi \in C_0^+(G)$, we get

$$\int_G \left\| \frac{1}{t} [f(y \cdot \exp(tX)) - f(y)] - \tilde{X}f(y) \right\|^2 \phi(y)\, dy$$

$$\leqslant \int_G \frac{1}{t} \int_0^t \| \tilde{X}f(y \cdot \exp(sX)) - \tilde{X}f(y)\|^2 \, ds\, \phi(y)\, dy$$

$$= \frac{1}{t} \int_0^t \int_G \| \tilde{X}f(y \cdot \exp(sX)) - \tilde{X}f(y)\|^2 \phi(y)\, dy\, ds.$$

By definition of the norm in $\mathbf{H}^L$ this gives

$$\left\| \frac{1}{t} [U^L(\exp(tX))f - f] - \tilde{X}f \right\|^2$$

$$\leqslant \frac{1}{t} \int_0^t \| U^L(\exp(sX))\, \tilde{X}f - \tilde{X}f \|^2 \, ds \to 0 \qquad \text{as} \qquad t \to 0,$$

since the integrand is a continuous function of s.

In other words, $\tilde{X}f$ is in the domain of the infinitesimal generator $u_1(X)$ of the one-parameter group $t \to U^L(\exp(tX))$ and $u_1(X)f = \tilde{X}f$. If $\tilde{X}^\alpha f \in \mathbf{H}^L$ for all α, it follows that f is in the domain of all powers of the operators $u_1(X)$, $X \in \mathfrak{g}$. Then, by Goodman's theorem we have $f \in \mathbf{D}_\infty$ and $u(D)f = \tilde{D}f$ for all $D \in \mathfrak{U}(\mathfrak{g})$ (For $D \in \mathfrak{U}(\mathfrak{g})$, we let $\tilde{D}$ denote the corresponding left invariant differential operator on G.)

In order to prove the other inclusion we introduce the functions $\epsilon(\phi, v)$, $\phi \in C_0^\infty(G)$ and $v \in \mathbf{V}$ [3, p. 82] defined as follows:

$$\epsilon(\phi, v)(x) = \int_H \phi(\xi \cdot x)\, \delta(\xi)^{-1/2} \Delta(\xi)^{1/2} L(\xi^{-1})v\, d\xi,$$

and we let $\mathbf{D} = \operatorname{span}\{\epsilon(\phi, v) \mid \phi \in C_0^\infty(G), v \in \mathbf{V}\}$.

Then $\mathbf{D} \subseteq \mathbf{D}_\infty$ [3, Lemma 6] and clearly $\mathbf{D}$ is invariant under the $U^L(g)$, $g \in G$. On the other hand $\mathbf{D}$ is dense in $\mathbf{H}^L$ [3, Lemma 2], so by Theorem 1.3 we get that $\mathbf{D}$ is dense in $\mathbf{D}_\infty$.

It is easy to see that $\epsilon(\phi, v) \in C^\infty(G, \mathbf{V})$ for all $\phi \in C_0^\infty(G)$ and $v \in \mathbf{V}$, so all functions in $\mathbf{D}$ are infinitely differentiable. To complete the proof we need the following result (which obviously is not the best possible but good enough for our present purposes):

LEMMA 5.2. *For each compact subset $K \subseteq G$ there exists a constant C_K such that*

$$\sup_{x \in K} \|f(x)\| \leqslant C_K \sum_{|\alpha| \leqslant d} \| \tilde{X}^\alpha f \|$$

for all $f \in \mathbf{D}$.

The proof will be given later.

Let $f \in \mathbf{D}_\infty(U^L)$. Then there exists a sequence $\{f_n\} \subseteq \mathbf{D}$ such that for all $D \in \mathfrak{U}(\mathfrak{g})$:

$$\| \tilde{D} f_n - u(D) f \| \to 0 \qquad \text{as} \qquad n \to \infty.$$

It follows from the proof of Proposition 1 in [3] that for each $D \in \mathfrak{U}(\mathfrak{g})$ there exists a subsequence $\{f_{n_k}\}$ (depending on D) such that $\tilde{D} f_{n_k} \to u(D) f$ l.a.e.

Let $\mathscr{E}(G, \mathbf{V})$ denote $C^\infty(G, \mathbf{V})$ as a topological vector space (with the usual topology). By Lemma 5.2 $\{f_n\}$ is a Cauchy sequence in $\mathscr{E}(G, \mathbf{V})$; so there exists a unique function $f_0 \in \mathscr{E}(G, \mathbf{V})$ such that $f_n \to f_0$; i.e., $\tilde{D} f_n \to \tilde{D} f$ uniformly on compact sets for all $D \in \mathfrak{U}(\mathfrak{g})$. Then $\tilde{D} f_0 = U(D) f$ l.a.e.; so we may assume $f = f_0$. Hence, f is infinitely differentiable and $\tilde{D} f = u(D) f$ for all $D \in \mathfrak{U}(\mathfrak{g})$.

Proof of Lemma 5.2. Let U be a compact neighborhood of e in G. By Lemma 5.1 there exists a constant C such that

$$\| f(e) \| \leqslant C \sum_{|\alpha| \leqslant d} \left\{ \int_U \| \tilde{X}^\alpha f(y) \|^2 \, dy \right\}^{1/2}$$

for all differentiable functions f from G to $\mathbf{V}$. Let $\phi \in C_0^+(G)$ such that $\phi = 1$ on U. Then

$$\| f(e) \| \leqslant C \sum_{|\alpha| \leqslant d} \left\{ \int_G \| \tilde{X}^\alpha f(y) \|^2 \phi(y) \, dy \right\}^{1/2}.$$

By the left invariance of $\tilde{X}^\alpha$ we get

$$\begin{aligned} \| f(x) \| &\leqslant C \sum_{|\alpha| \leqslant d} \left\{ \int_G \| \tilde{X}^\alpha f(x \cdot y) \|^2 \phi(y) \, dy \right\}^{1/2} \\ &= \Delta(x)^{-1/2} C \sum_{|\alpha| \leqslant d} \left\{ \int_G \| \tilde{X}^\alpha f(y) \|^2 \phi(x^{-1} y) \, dy \right\}^{1/2}. \end{aligned}$$

If $\tilde{X}^\alpha f \in \mathbf{H}^L$ for all α with $|\alpha| \leqslant d$ this gives

$$\| f(x) \| \leqslant \Delta(x)^{-1/2} C \sum_{|\alpha| \leqslant d} \| \tilde{X}^\alpha f \| \cdot \| \tau(L(x)\phi) \|_\infty^{1/2}$$

for all $x \in G$, where $(L(x)\phi)(y) = \phi(x^{-1}y)$. Because L is a continuous representation of G in $C_0(G)$ and τ is a continuous linear mapping from $C_0(G)$ to $C_0(M)$ (see [4, Chapter VII, Section 2]), it follows that the functions $\tau(L(x)\phi)$, $x \in K$, are uniformly bounded. Since Δ^{-1} is bounded on K this completes the proof. Q.E.D.

COROLLARY 5.1. *For each compact subset $K \subseteq G$ there exists a constant C_K such that*

$$\|f(x)\| \leqslant C_K \sum_{|\alpha| \leqslant d} \|\tilde{X}^\alpha f\|$$

for all $x \in K$, $f \in \mathbf{D}_\infty(U^L)$.

In particular, for each fixed $x \in G$, $f \to f(x)$ is a continuous linear mapping from $\mathbf{D}_\infty$ to $\mathbf{V}$.

COROLLARY 5.2. *If f is an analytic vector for U^L, then f is an analytic function from G to $\mathbf{V}$.*

Proof. By Nelson's characterization of the analytic vectors (cf. Section 1) there exists a constant $C > 0$ such that

$$\| \tilde{X}_{i_1} \cdots \tilde{X}_{i_n} f \| \leqslant C^n \cdot n! \qquad \text{for all } n.$$

Let $K \subseteq G$ be any compact set. By Corollary 5.1 we have

$$\begin{aligned} \sup_{x \in K} \| \tilde{X}^\alpha f(x)\| &\leqslant \text{const} \sum_{|\beta| \leqslant d} \| \tilde{X}^\beta \tilde{X}^\alpha f \| \\ &\leqslant \text{const} \sum_{|\beta| \leqslant d} C^{|\alpha+\beta|} \, | \alpha + \beta |! \qquad \text{for all } \alpha. \end{aligned}$$

Therefore there exists a constant $C_K > 0$ such that

$$\sup_{x \in K} \| \tilde{X}^\alpha f(x)\| \leqslant C_K^{|\alpha|} \, | \alpha |! \qquad \text{for all } \alpha.$$

This completes the proof (see e.g. [1] or [19, Theorem 2]). Q.E.D.

Remark. The "converse" is not true. There exists an analytic function f on $\mathbb{R}$ such that $f^{(n)} \in L^2(\mathbb{R})$ for $n = 0, 1, 2, \ldots$, but f is not an analytic vector for the regular representation. (For example one can show that the function F given in [14, p. 177] has these properties.)

Remark. There seems to be a gap in the proof of Lemma 5.3 in [21]. The present Lemma 5.2 and Corollary 5.1 should be substituted for the corresponding statements in [21].

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